C. R. Acad. Sci. Paris, t. 326, Série I, p. 43-48, 1998
Contrôle optimal/*Optimal Control*

The duality theorem for min-max functions

Stéphane GAUBERT [a], **Jeremy GUNAWARDENA** [b]

[a] INRIA, domaine de Voluceau, B.P. 105, 78153 Le Chesnay cedex, France
E-mail: Stephane.Gaubert@inria.fr

[b] BRIMS, Hewlett-Packard Labs, Filton Road, Stoke Gifford, Bristol BS12 6QZ, UK
E-mail: jhcg@hplb.hpl.hp.com

(Reçu le 12 juillet 1997, accepté après révision le 27 novembre 1997)

Abstract. The set of min-max functions $F : \mathbb{R}^n \to \mathbb{R}^n$ is the least set containing coordinate substitutions and translations and closed under pointwise max, min, and function composition. The Duality Conjecture asserts that the trajectories of a min-max function, considered as a dynamical system, have a linear growth rate (cycle time) and shows how this can be calculated through a representation of F as an infimum of max-plus linear functions. We prove the conjecture using an analogue of Howard's policy improvement scheme, carried out in a lattice ordered group of germs of affine functions at infinity. The methods yield an efficient algorithm for computing cycle times.

Le théorème de dualité pour les fonctions min-max

Résumé. *L'ensemble des fonctions min-max $F : \mathbb{R}^n \to \mathbb{R}^n$ est le plus petit ensemble de fonctions qui contient les substitutions de coordonnées et les translations, et qui est stable par les opérations* min *et* max *(point par point), ainsi que par composition. La conjecture de dualité affirme que les trajectoires d'un système récurrent gouverné par une dynamique min-max ont un taux de croissance linéaire (temps de cycle), qui se calcule à partir d'une représentation de F comme infimum de fonctions max-plus linéaires. Nous montrons cette conjecture en utilisant une itération sur les politiques à la Howard, à valeurs dans un groupe réticulé de germes de fonctions affines à l'infini. On a ainsi un algorithme efficace pour calculer le temps de cycle.*

Version française abrégée

Nous munissons $\mathbb{R}^n$ et l'ensemble des fonctions $\mathbb{R}^n \to \mathbb{R}^n$ de l'ordre partiel usuel (composante par composante). Les bornes sup et inf sont notées $\vee$ et $\wedge$, respectivement. Nous appellerons *substitution* une application $F : \mathbb{R}^n \to \mathbb{R}^n$ de la forme $F(x)_i = x_{\pi(i)}$, où π est une transformation quelconque de $\{1, \ldots, n\}$ (peut-être non bijective). Une *translation* est une application de la forme: $\mathbb{R}^n \to \mathbb{R}^n$, $x \mapsto x + u$, avec $u \in \mathbb{R}^n$. L'ensemble des fonctions min-max est le plus petit ensemble de fonctions $\mathbb{R}^n \to \mathbb{R}^n$ contenant les substitutions et les translations, et qui est stable par les opérations binaires $\vee, \wedge$ et la composition.

Note présentée par Pierre-Louis LIONS.

Les fonctions min-max comprennent les applications max-plus linéaires, (*voir* [4], [1], [15] et [7]), qui sont de la forme $G(x)_i = \max_{1 \leq j \leq n}(A_{ij} + x_j)$, A étant une matrice $n \times n$ à coefficients dans $\mathbb{R} \cup \{-\infty\}$, avec au moins un coefficient fini par ligne. Elles comprennent aussi les applications min-plus linéaires, définies dualement.

Comme les opérations $\vee$ et $\wedge$ distribuent l'une par rapport à l'autre, nous pouvons écrire toute fonction min-max sous la forme (1), où $\mathcal{G}$ et $\mathcal{H}$ sont des ensembles finis d'applications max-plus et min-plus linéaires, respectivement.

On appelle vecteur de *temps de cycle* de F la quantité (2), avec $\xi(k) = F(\xi(k-1))$ pour $k \geq 1$ et $\xi(0) = x \in \mathbb{R}^n$. Quand elle existe, cette limite ne dépend pas du point initial x, car les fonctions min-max sont non dilatantes (c'est-à-dire Lipschitz de constante 1) pour la norme-sup. Pour les applications max-plus ou min-plus linéaires, l'existence et le calcul du temps de cycle sont explicités dans [8]. Cela résulte de l'analogue max-plus de la théorie de Perron–Frobenius (*voir* par exemple [1], §3.2.4, §3.7, [15] et [7] § 3.7).

Un ensemble $\mathcal{S}$ de fonctions min-max est *rectangulaire* si pour tous $G, G' \in \mathcal{S}$, et pour tout $i = 1, \ldots, n$, la fonction obtenue en remplaçant la i-ème ligne de G par celle de G' appartient encore à $\mathcal{S}$. Nous noterons $\overline{\mathcal{S}}$ la *clôture rectangulaire* (plus petit sur-ensemble rectangulaire) de $\mathcal{S}$. Évidemment, on peut prendre le sup ou l'inf sur la clôture rectangulaire sans changer la valeur de F, ce qui donne (3). Comme les fonctions min-max sont monotones, on déduit aisément (4) de (3), pour tout point d'accumulation $\overline{\chi}$ de la suite $\frac{1}{k} \times \xi(k)$. La *conjecture de dualité*, énoncée par Gunawardena dans [8], affirme que les membres extrêmes de (4) sont égaux. Elle entraîne l'existence du temps de cycle $\chi(F)$.

Nous définissons la relation d'équivalence sur $\mathbb{R}^{\mathbb{N}}$, $f \sim g \iff \exists K \in \mathbb{N}, \forall k \geq K, f(k) = g(k)$. Nous notons $\mathbb{G}$ l'ensemble des germes de fonctions affine en $+\infty$, c'est-à-dire l'image de l'ensemble des fonctions affines par la projection canonique : $\mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}/\sim$. Les lois usuelles $\vee, \wedge, +, \times$ sur $\mathbb{R}^{\mathbb{N}}$ définissent des lois quotients sur $\mathbb{R}^{\mathbb{N}}/\sim$. En particulier, $(\mathbb{G}, \vee, \wedge, +)$ est un groupe (totalement) ordonné en treillis. Nous notons ω la classe d'équivalence de l'injection $\mathbb{N} \to \mathbb{R}, k \mapsto k$. Un germe $u \in \mathbb{G}$ s'écrit de façon unique $u = a + \alpha\omega$, où $a, \alpha \in \mathbb{R}$, et $\ell(u) \stackrel{\text{déf}}{=} \alpha$ est la *partie linéaire* de u. Ainsi, $((3+2\omega) \vee (2+5\omega)) \wedge (-100+7\omega) = 2+5\omega$, car $((3+2k) \vee (2+5k)) \wedge (-100+7k) = 2+5k$, quand k est assez grand. Le décalage $\theta : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$, $\theta u(k) = u(k+1)$ induit une application $\theta : \mathbb{G} \to \mathbb{G}$, $\theta(a + \alpha\omega) = a + \alpha + \alpha\omega$. Nous entendrons les notations $\theta, \ell, \vee, \wedge, +$ aux vecteurs, composante par composante. Une fonction min-max F définit une application $\mathbb{G}^n \to \mathbb{G}^n$, $u \mapsto F \circ u$, que nous noterons encore F. Un *vecteur propre généralisé* de F est un vecteur $u \in \mathbb{G}^n$ tel que $F(u) = \theta u$. Si $u = a + \alpha\omega$, avec $a, \alpha \in \mathbb{R}^n$, cela s'écrit comme en (5).

THÉORÈME. – *Toute fonction min-max admet un vecteur propre généralisé.*

COROLLAIRE. – *Toute fonction min-max admet un temps de cycle.*

COROLLAIRE. – *La conjecture de dualité est satisfaite.*

La preuve du théorème, donnée dans [6], repose sur un argument d'amélioration des politiques à la Howard. Elle fournit un algorithme efficace pour calculer le temps de cycle. La preuve de convergence utilise un principe du maximum max-plus.

1. Min-max functions

We equip $\mathbb{R}^n$ and functions $\mathbb{R}^n \to \mathbb{R}^n$ with the usual (pointwise) partial order. The least upper bound and greatest lower bound are denoted $\vee$ and $\wedge$, respectively. We call *substitution* a function

$F : \mathbb{R}^n \to \mathbb{R}^n$ of the form $F(x)_i = x_{\pi(i)}$, where π is an arbitrary (possibly non-bijective) transformation of $\{1, \dots, n\}$. A *translation* is a map $\mathbb{R}^n \to \mathbb{R}^n$, $x \mapsto x + u$, with $u \in \mathbb{R}^n$. The set of min-max functions is the least set of functions $\mathbb{R}^n \to \mathbb{R}^n$, which contains substitutions and translations, and is stable by the binary operations $\vee, \wedge$ and function composition.

Min-max functions include max-plus linear maps (i.e. linear maps over the max-plus semiring *see* [4], [1], [15], [7]), which can be written as $G(x)_i = \max_{1 \le j \le n}(A_{ij} + x_j)$, where A is a $n \times n$ matrix with entries in $\mathbb{R} \cup \{-\infty\}$, with at least one finite entry per row. Min-plus linear maps, defined dually, are also special min-max functions.

Using the mutual distributivity of $\vee$ and $\wedge$, we can write any min-max function as

$$F(x) = \bigwedge_{G \in \mathcal{G}} G(x) = \bigvee_{H \in \mathcal{H}} H(x) \qquad (1)$$

where $\mathcal{G}$ and $\mathcal{H}$ are finite sets of max-plus and min-plus linear maps, respectively.

Clearly, a min-max function F is *non-expansive* for the sup-norm $\| \cdot \|$ (i.e. $\|F(x) - F(y)\| \le \|x - y\|$), it is *monotone* (i.e. $x \le y \Rightarrow F(x) \le F(y)$), and (additively) *homogeneous* (i.e. $F(x_1 + \alpha, \dots, x_n + \alpha)_i = F(x_1, \dots, x_n)_i + \alpha$, for all $i = 1, \dots, n$).

Consider the trajectory $\xi(k) = F(\xi(k-1))$ for $k \ge 1$, $\xi(0) = x \in \mathbb{R}^n$. The *cycle-time* vector of F, $\chi(F)$, is defined by:

$$\chi(F) = \lim_{k \to \infty} \frac{1}{k} \times \xi(k) \qquad (2)$$

Since F is nonexpansive, the limit, if it exists, is independent of the initial point x. When F is a max-plus linear map (or dually, a min-plus linear map), the existence of $\chi(F)$ follows from the max-plus analogue of the Perron–Frobenius theorem (*see* e.g. [1], [5], and [7] for recent presentations of this ancient result). Indeed, if the graph canonically associated to F is strongly connected, F has a finite max-plus eigenvector, i.e. $F(u) = (\lambda + u_1, \dots, \lambda + u_n)$, with $\lambda \in \mathbb{R}, u \in \mathbb{R}^n$. Then, $\chi(F) = (\lambda, \dots, \lambda)$. For a general max-plus linear map, the cycle time can be determined from the partition of the associated graph in strongly connected components, as detailed in [8], Prop. 7. In this case, F need not admit a finite eigenvector, but it does admit a generalized eigenvector, as defined in § 2 below.

A set $\mathcal{S}$ of min-max functions is *rectangular* if for all $G, G' \in \mathcal{S}$, and for all $i = 1, \dots, n$, the function obtained by replacing the i-th component of G by the i-th component of G' belongs to $\mathcal{S}$. We denote by $\overline{\mathcal{S}}$ the *rectangular closure* (least rectangular supset) of a set $\mathcal{S}$, which is finite when $\mathcal{S}$ is finite. The min-max function (1) can be written equivalently as:

$$F(x) = \bigwedge_{G \in \overline{\mathcal{G}}} G(x) = \min_{G \in \overline{\mathcal{G}}} G(x) = \bigvee_{H \in \overline{\mathcal{H}}} H(x) = \max_{H \in \overline{\mathcal{H}}} H(x) \qquad (3)$$

where "min" and "max" mean that the greatest lower bound and least upper bound are attained. Since min-max functions are monotone, we infer from (3) that

$$\bigwedge_{G \in \overline{\mathcal{G}}} \chi(G) \ge \overline{\chi} \ge \bigvee_{H \in \overline{\mathcal{H}}} \chi(H) \qquad (4)$$

for any accumulation point $\overline{\chi}$ of the sequence $\frac{1}{k} \times \xi(k)$. The *Duality Conjecture*, proposed by Gunawardena in [8], states that the extreme sides in (4) coincide. Hence, $\chi(F) = \overline{\chi}$ exists, and its value is given explicitly by (4).

The duality conjecture was proved for $n = 2$ in [2]. The cycle-time vector was shown to exist for $n = 3$ in [18]. The purpose of this Note is to prove the duality conjecture, and to show how the cycle time can be computed efficiently.

Min-max functions were introduced in [9] following earlier work on special cases in [17]. Functions which are homogeneous and nonexpansive in the supremum norm —so called *topical functions*— have appeared in the work of several authors, in particular [3], [16], [11], and [14]. They include (after suitable transformation) nonnegative matrices, Leontieff substitution systems, Bellman operators of games and of Markov decision processes with average cost and dynamical models of discrete event systems (digital circuits, computer networks, manufacturing plants, etc.). Min-max functions play an important role within this larger class because any topical function T can be approximated by min-max functions in such a way that some of the dynamical behaviour of T is inherited by its approximations (*see* [12]). The cycle-time vector appears in this context as a nonlinear generalization of the conventional spectral radius of a nonnegative matrix. However, it does not always exist (*see* [11]), and it is an important open problem to determine those topical functions for which it does. The positive solution of the duality conjecture shows that it does exist for min-max functions. In earlier work [2], we also showed that the duality conjecture implied the following strong fixed point theorem: $F(x_1, \ldots, x_n) = (x_1 + \alpha, \ldots, x_n + \alpha)$ if and only if $\chi(F) = (\alpha, \ldots, \alpha)$, which subsumes earlier results in the literature, such as [17]. The results of the present paper provide an independent proof of this fixed point theorem.

The algorithm below is an extension to germs of the policy iteration algorithm given in [2], for the special min-max functions whose policies have an additive eigenvector. Classical policy iteration algorithms à la Howard (for stochastic control) can be found e.g. in [19].

2. Generalized eigenvectors and germs of affine functions

We define an equivalence relation on $\mathbb{R}^{\mathbb{N}}$ by $f \sim g \iff \exists K \in \mathbb{N}, \forall k \geq K, f(k) = g(k)$. A map $f \in \mathbb{R}^{\mathbb{N}}$ is *affine* if $f(k) = a + \alpha k$, with $a, \alpha \in \mathbb{R}$. We denote by $\mathbb{G}$ the image of the set of affine functions by the canonical projection: $\mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}/\sim$. An element of $\mathbb{G}$ is a *germ of affine functions at* $+\infty$. The usual laws $\vee, \wedge, +, \times$ on $\mathbb{R}^{\mathbb{N}}$ induce quotient laws on $\mathbb{R}^{\mathbb{N}}/\sim$. In particular, $(\mathbb{G}, \vee, \wedge, +)$ is a lattice (totally) ordered group. We denote by ω the equivalence class of the injection $\mathbb{N} \to \mathbb{R}, k \mapsto k$. An element u of $\mathbb{G}$ can be written in a unique way as $u = a + \alpha\omega$, with $a, \alpha \in \mathbb{R}$. We call $\ell(u) \stackrel{\text{def}}{=} \alpha$ the *linear part* of u. *E.g.* $((3 + 2\omega) \vee (2 + 5\omega)) \wedge (-100 + 7\omega) = 2 + 5\omega$, for $((3 + 2k) \vee (2 + 5k)) \wedge (-100 + 7k) = 2 + 5k$, for k large enough. The shift operator $\theta : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$, $\theta u(k) = u(k + 1)$ induces a map $\theta : \mathbb{G} \to \mathbb{G}$, $\theta(a + \alpha\omega) = a + \alpha + \alpha\omega$. The notation $\theta, \ell, \vee, \wedge, +$ will be extended to vectors (entrywise). A min-max function F defines a map (also denoted F) $\mathbb{G}^n \to \mathbb{G}^n$, that to the equivalence class of u, associates the equivalence class of $F \circ u$.

A *generalized eigenvector* of F is a vector $u \in \mathbb{G}^n$ such that $F(u) = \theta u$. More explicitly, if $u = a + \alpha\omega$, with $a, \alpha \in \mathbb{R}^n$,

$$\exists K, \forall k \geq K, \ F(a + k\alpha) = a + (k + 1)\alpha \tag{5}$$

When $\alpha = (\lambda, \ldots, \lambda)$ with $\lambda \in \mathbb{R}$, a is a (nonlinear, additive) eigenvector of F, with (nonlinear, additive) eigenvalue λ.

THEOREM 1. – *Any min-max function has a generalized eigenvector.*

Before sketching the proof technique, we mention two immediate consequences.

COROLLARY 1. – *Any min-max function has a cycle time.*

Proof. – Let $u = a + \alpha\omega$ denote a generalized eigenvector of a min-max function F. By (5), setting $\xi(0) = a + K\alpha$, we obtain $\xi(k) = F(\xi(k-1)) = a + (k+K)\alpha$. Hence, $\chi(F) = \ell(u) = \alpha$.

COROLLARY 2. – *The duality conjecture holds.*

Proof. – Let u denote a generalized eigenvector of the min-max function F given by (1). Since $\mathbb{G}$ is totally ordered and $\overline{\mathcal{G}}$ is rectangular, $\theta u = F(u) = G(u)$ for some $G \in \overline{\mathcal{G}}$. Hence, $\chi(F) = \chi(G) = \ell(u)$. This shows that the first relation in (4) is an equality. By symmetry, the second relation is also an equality, and the duality conjecture holds.

The proof of Theorem 1, given in [6], relies on a policy iteration argument, which is analogous to Howard's multichain policy iteration algorithm for Markov decision processes with average cost (*see* e.g. [19]). Whereas the conventional policy iteration algorithm solves essentially a sequence of generalized spectral problems of the form $c^{(k)} + P^{(k)}u = \theta u$, with $u \in \mathbb{G}^n$ (at step k, $c^{(k)} \in \mathbb{R}^n$ is given, and $P^{(k)}$ is a given $n \times n$ row-stochastic matrix), the min-max policy iteration algorithm below solves a sequence of generalized spectral problems which are linear over the max-plus semiring. The proof that it terminates is based on the following max-plus analogue of the maximum principle.

LEMMA 1 (Max-plus maximum principle). – *Let G denote a max-plus linear map $\mathbb{R}^n \to \mathbb{R}^n$ such that $\chi(G) = 0$. Let $z \in \mathbb{R}^n$ such that $G(z) \leq z$. Let C denote the set of nodes of the* critical graph [1], §3.2.4, §3.7, [7], § 3.7, *of G. There exists a unique $x \in \mathbb{R}^n$ which is a solution of the "Dirichlet problem"*

$$G(x)_i = x_i \quad \forall i \notin C, \quad \textit{and} \quad x_i = z_i \quad \forall i \in C$$

Furthermore, $x \leq z$.

If G is a max-plus linear map, we set $\widetilde{G} = -\chi(G) + G$. The *spectral projector* P_G is defined by $\mathrm{P}_G(x) = \limsup_{k\to\infty} \widetilde{G}^k(x)$. When z and x are as in Lemma 1, we can check that $x = \mathrm{P}_G(z)$. Hence, the max-plus maximum principle implies that $u^{(k+1)} \leq u^{(k)}$ at each type (4b) step of Algorithm 1 below. The cycle time $\chi(G)$, the spectral projector, and a generalized eigenvector of G, can be computed in $O(n^3)$ time using Karp's algorithm (*see* [13] and [1], § 3.2.4, § 3.7). Consider F as in (1). We will call *policies* the maps $G \in \overline{\mathcal{G}}$. The i-th component of a policy G is denoted G_i.

ALGORITHM 1 (Min-max policy iteration). – *Input:* A min-max function F. *Output:* $u \in \mathbb{G}^n$ such that $F(u) = \theta u$.

1. *Initialization:* Select an arbitrary policy $G^{(1)} \in \overline{\mathcal{G}}$. Set $k = 1$. Find $u^{(1)} \in \mathbb{G}^n$ such that $G^{(1)}u^{(1)} = \theta u^{(1)}$.

2. Let

$$I_k = \{i \in \{1,\dots,n\} \mid F(u^{(k)})_i < (\theta u^{(k)})_i\}, \qquad J_k = \{i \in \{1,\dots,n\} \mid \ell(F(u^{(k)}))_i < \ell(u^{(k)})_i\}$$

If $I_k = \emptyset$, we have $F(u^{(k)}) = \theta u^{(k)}$. Stop.

3. *Policy improvement:* We select $G' \in \overline{\mathcal{G}}$ such that $G'(u^{(k)}) = F(u^{(k)})$.
(a) If $J_k \neq \emptyset$, we set: $G_i^{(k+1)} = G'_i$ if $i \in J_k$, $G_i^{(k)}$ if $i \notin J_k$.
(b) If $J_k = \emptyset$, we set: $G_i^{(k+1)} = G'_i$ if $i \in I_k$, $G_i^{(k)}$ if $i \notin I_k$.

4. *Value determination:* Find $v \in \mathbb{G}^n$ such that $G^{(k+1)}(v) = \theta v$.
(a) If $\ell(v) \neq \ell(u^{(k)})$, then set $u^{(k+1)} = v$.
(b) If $\ell(v) = \ell(u^{(k)})$, then select $u^{(k+1)} = \mathrm{P}_{G^{(k+1)}}u^{(k)}$.

5. Increment k by one and go to step 2.

Example 1. – We apply Algorithm 1 to:

$$F : \mathbb{R}^3 \to \mathbb{R}^3,\ F(x) = \begin{pmatrix} ((x_2+2) \vee (x_3+5)) \wedge x_1 \\ (x_2+1) \wedge (x_3+2) \\ (x_1-1) \vee (x_2+3) \end{pmatrix}$$

We start from

$$G^{(1)}(x) = \begin{pmatrix} (x_2+2) \vee (x_3+5) \\ x_3+2 \\ (x_1-1) \vee (x_2+3) \end{pmatrix}, \qquad u^{(1)} = \begin{pmatrix} 3+2.5\omega \\ 2.5\omega \\ 0.5+2.5\omega \end{pmatrix}$$

This is a type (3b) policy improvement: we get $F(u^{(1)}) = G^{(2)}(u^{(1)})$ with

$$G^{(2)}(x) = \begin{pmatrix} x_1 \\ x_2+1 \\ (x_1-1) \vee (x_2+3) \end{pmatrix}, \qquad u^{(2)} = \begin{pmatrix} 0 \\ 1\omega \\ 2+1\omega \end{pmatrix}$$

Then the algorithm terminates since $F(u^{(2)}) = \theta u^{(2)}$. Thus, $\chi(F) = \ell(u^{(2)}) = (0, 1, 1)$.

To conclude, we propose an extension of the duality theorem. We will call min-max-$\mathbb{E}$ *function* a map of the form $F(x) = \bigwedge_{i \in I} \bigvee_{j \in J} (c^{ij} + P^{ij}x)$, where I, J are finite sets, and for all $(i,j) \in I \times J$, $c^{ij} \in \mathbb{R}^n$ and P^{ij} is a row-stochastic matrix. When the P^{ij} matrices have exactly one nonzero entry per row, we obtain min-max functions. Using a vanishing discount argument, one can prove that a min-max-$\mathbb{E}$ function admits a generalized eigenvector. This will be detailed elsewhere.

References

[1] F. Baccelli, G. Cohen, G.J. Olsder, J.P. Quadrat. Synchronization and Linearity. Wiley, 1992.
[2] J. Cochet-Terrasson, S. Gaubert, J. Gunawardena. Dynamics of min-max functions. Submitted for publication. Also Technical Report HPL-BRIMS-97-13, 1997.
[3] M.G. Crandall, L. Tartar. Some relations between non expansive and order preserving maps. Proc. Amer. Math. Soc., 78(3) (1980), 385–390.
[4] R.A. Cuninghame-Green. Minimax Algebra. Lecture Notes in Economics and Mathematical Systems, 166, Springer, 1979.
[5] R.A. Cuninghame-Green. Minimax algebra and applications. Advances in Imaging and Electron Physics, 90, 1995.
[6] S. Gaubert, J. Gunawardena. A proof of the duality conjecture for min-max functions. Manuscript attached to this note, 1997.
[7] S. Gaubert, M. Plus. Methods and applications of (max,+) linear algebra. In: R. Reischuk, M. Morvan, editors, STACS'97, LNCS, 1200, Lübeck, March 1997. Springer.
[8] J. Gunawardena. Cycle time and fixed points of min-max functions. In: G. Cohen, J.P. Quadrat, editors, 11th International Conference on Analysis and Optimization of Systems, LNCIS, 199, 266–272. Springer, 1994.
[9] J. Gunawardena. Min-max functions. J. Discrete Event Dynamic Systems, 4 (1994), 377–406.
[10] J. Gunawardena, editor. Idempotency. Publications of the Newton Institute. Cambridge University Press, 1997.
[11] J. Gunawardena, M. Keane. On the existence of cycle times for some nonexpansive maps. Technical Report HPL-BRIMS-95-003, Hewlett-Packard Labs, 1995.
[12] J. Gunawardena, M. Keane, C. Sparrow. In preparation, 1997.
[13] R.M. Karp. A characterization of the minimum mean-cycle in a digraph. Discrete Maths., 23 (1978), 309–311.
[14] V. Kolokoltsov. Linear additive and homogeneous operators. Appears in [15].
[15] V. Maslov, S. Samborskiĭ, editors. Idempotent analysis, Adv. Sov. Math. AMS, 13, RI, 1992.
[16] R.D. Nussbaum. Hilbert's projective metric and iterated nonlinear maps. Memoirs of the AMS, 95(391), 1990.
[17] G.J. Olsder. Eigenvalues of dynamic max-min systems. J. Discrete Event Dynamic Systems, 1, 177–207, 1991.
[18] C. Sparrow. Existence of cycle time for minmax functions of dimension 3. Technical Report HPL-BRIMS-96-008, Hewlett-Packard Labs. 1996.
[19] P. Whittle. Optimization over Time, Wiley, 1986.