On the existence of cycle times for some non-expansive maps

Jeremy Gunawardena
BRIMS, Hewlett-Packard Labs
Filton Road, Stoke Gifford
Bristol BS12 6QZ, UK
jhcg@hplb.hpl.hp.com

Mike Keane
CWI, PO Box 94079
1090 GB Amsterdam
The Netherlands
keane@cwi.nl

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Abstract

We consider functions $F : \mathbf{R}^n \to \mathbf{R}^n$ which are homogeneous and nonexpansive in the ℓ^∞ norm. We refer to these as topical functions. We study the existence of the cycle time vector $\chi(F) = \lim_{k\to\infty} F^k(\vec{x})/k$, which, if it exists, is independent of $\vec{x} \in \mathbf{R}^n$. For a restricted class of topical functions, the cycle time is known to be implicated in the existence of fixed points and this provides the motivation for the present paper. We give a characterisation of topical functions which extends an earlier result of Crandall and Tartar. We show that the sequence $F^k(\vec{x})/k$ converges weakly, in the sense that its images under the functions $\mathsf{t}(x_1, \cdots, x_n) = \max\{x_1, \cdots, x_n\}$ and $\mathsf{b}(x_1, \cdots, x_n) = \min\{x_1, \cdots, x_n\}$ always converge. We show that under suitable conditions, weak convergence may be realised by the convergence of components of the vector sequence $F^k(0)/k$. We show further that when $n = 2$, χ itself exists. When $n = 3$, it may not, and we give a family of examples which show the extent of the departure from convergence. We discuss the problem of characterising those topical functions for which χ does exist.

1 Elementary properties

We begin by collecting together some of the notation that we shall use. We then characterise the class of functions which we shall study.

Vectors in $\mathbf{R}^n$ will be denoted by $\vec{x}$, $\vec{y}$, etc and x_i will denote the i-th component of $\vec{x}$. We use the notation $\vec{x} \le \vec{y}$ for the partial order on $\mathbf{R}^n$ coming from the product of the partial orders on each component: $x_i \le y_i$ for $1 \le i \le n$.

The notation $a \vee b$ and $a \wedge b$ will stand for maximum (least upper bound) and minimum (greatest lower bound) respectively of real numbers: $a \vee b = \max(a, b)$ and $a \wedge b = \min(a, b)$. Note that addition distributes over both maximum and minimum:

$$h + (a \vee b) = h + a \vee h + b, \qquad h + (a \wedge b) = h + a \wedge h + b. \tag{1}$$

The same symbols will also be used for the corresponding operations on vectors. Since the ordering is the product ordering, it is easy to see that

$$\begin{aligned} (\vec{x} \vee \vec{y})_i &= x_i \vee y_i \\ (\vec{x} \wedge \vec{y})_i &= x_i \wedge y_i \end{aligned}$$

It will be convenient to use the following convention for formulae involving both vectors and scalars. When scalars and vectors appear in the same formula, the scalar operation or relation is performed on each component of the vector. So, for instance, $\vec{x} = h$ means $x_i = h$ for each $1 \le i \le n$. Similarly, $\vec{x} + h$ is $\vec{x}$ with h added to each component. This "vector-scalar" convention keeps additional notation to a minimum.

If $\vec{x} \in \mathbf{R}^n$, then the top of $\vec{x}$, $\mathsf{t}(\vec{x})$, and the bottom of $\vec{x}$, $\mathsf{b}(\vec{x})$, are defined as follows:

$$\begin{aligned} \mathsf{t}(\vec{x}) &= x_1 \vee \cdots \vee x_n \\ \mathsf{b}(\vec{x}) &= x_1 \wedge \cdots \wedge x_n \ . \end{aligned}$$

The following trivial formulae involving t and b will be helpful:

$$\begin{aligned} \mathsf{b}(\vec{x}) &= -\mathsf{t}(-\vec{x}) \\ \vec{x} &\le \mathsf{t}(\vec{x}) \\ \mathsf{t}(\lambda\vec{x}) &= \lambda\mathsf{t}(\vec{x}) \quad \text{if } \lambda \ge 0 \\ \mathsf{t}(\vec{x}) &\le \mathsf{t}(\vec{y}) \quad \text{if } \vec{x} \le \vec{y} \\ \mathsf{t}(\vec{x} + h) &= \mathsf{t}(\vec{x}) + h \\ \mathsf{t}(\vec{x} + \vec{y}) &\le \mathsf{t}(\vec{x}) + \mathsf{t}(\vec{y}) \\ \mathsf{t}(\vec{x} \vee \vec{y}) &= \mathsf{t}(\vec{x}) \vee \mathsf{t}(\vec{y}) \\ \mathsf{t}(\vec{x} \wedge \vec{y}) &\le \mathsf{t}(\vec{x}) \wedge \mathsf{t}(\vec{y}) \\ \|\vec{x}\| &= \mathsf{t}(\vec{x}) \vee -\mathsf{b}(\vec{x}) \end{aligned} \tag{2}$$

where $\|\vec{x}\|$ denotes the ℓ^∞ norm on $\mathbf{R}^n$: $\|\vec{x}\| = \|x_1\| \vee \cdots \vee \|x_n\|$.

If $A \subseteq \mathbf{R}^n$ is a finite set of vectors let $\Box(A)$ denote the rectangularistion of A:

$$\Box(A) = \{\vec{u} \in \mathbf{R}^n \mid \forall u_i, \exists \vec{x} \in A, \text{ such that } u_i = x_i\}.$$

We can then note that

$$\mathsf{t}(\vec{x} \wedge \vec{y}) = \bigwedge_{\vec{u} \in \Box\{\vec{x}, \vec{y}\}} \vec{u}.$$

We shall be interested in functions $F : \mathbf{R}^n \to \mathbf{R}^n$ with the properties below, which we shall denote by the associated letters. Universal quantification over variables is to be understood in each of the statements.

- $F(\vec{x} + h) = F(\vec{x}) + h$ (homogeneity) H
- $\vec{x} \leq \vec{y} \implies F(\vec{x}) \leq F(\vec{y})$ (monotonicity) M
- $\|F(\vec{x}) - F(\vec{y})\| \leq \|\vec{x} - \vec{y}\|$ (nonexpansiveness in ℓ^∞) N
- $\mathsf{b}(F(\vec{x}) - F(\vec{y})) \geq \mathsf{b}(\vec{x} - \vec{y})$ (noncontractiveness in b) B
- $\mathsf{t}(F(\vec{x}) - F(\vec{y})) \leq \mathsf{t}(\vec{x} - \vec{y})$ (nonexpansiveness in t) T

Note that t is not even a seminorm, since $\mathsf{t}(\vec{x})$ may be negative. However, it is helpful to think of it in this way, as we shall see.

We can now describe the functions which we shall study in a number of different ways.

Proposition 1.1 *The following statements are equivalent.*

1. *F satisfies* H *and* M.
2. *F satisfies* H *and* N.
3. *F satisfies* B.
4. *F satisfies* T.

Proof: We shall show that the first three statements are each equivalent to the fourth.

($3 \Leftrightarrow 4$) Choose $\vec{x}, \vec{y} \in R^n$. By the first formula in (2),

$$\begin{aligned} \mathsf{b}(F(\vec{x}) - F(\vec{y})) &= -\mathsf{t}(F(\vec{y}) - F(\vec{x})) \\ \mathsf{b}(\vec{x} - \vec{y}) &= -\mathsf{t}(\vec{y} - \vec{x}) \end{aligned}$$

The equivalence of statements 3 and 4 is now evident.

($1 \Leftrightarrow 4$) Suppose statement 4 holds. Let $\vec{y} = \vec{x} + h$. Since $\mathsf{b}(\vec{y} - \vec{x}) = \mathsf{t}(\vec{y} - \vec{x}) = h$, it follows from T and B that $h \leq F(\vec{x}+h) - F(\vec{x}) \leq h$, from which H follows immediately. Now suppose that $\vec{x} \leq \vec{y}$. It follows that $0 \leq \mathsf{b}(\vec{y} - \vec{x})$ and so, by B, $0 \leq \mathsf{b}(F(\vec{y}) - F(\vec{x}))$. Hence $F(\vec{x}) \leq F(\vec{y})$, which demonstrates M. Hence statement 1 holds.

Now suppose statement 1 holds and choose $\vec{x}, \vec{y} \in R^n$. Since $\vec{x} \leq \vec{y} + \mathsf{t}(\vec{x} - \vec{y})$, by applying F to both sides and using H and M, we see that $F(\vec{x}) \leq F(\vec{y}) + \mathsf{t}(\vec{x} - \vec{y})$, from which statement 4 easily follows.

($2 \Leftrightarrow 4$) Suppose statement 4 holds. We have already shown that property H must then be true. By combining statements 3 and 4 using the last formula in (2), it is easy to see that N must also hold. Hence statement 2 holds.

Now suppose statement 2 holds and choose $\vec{x}, \vec{y} \in R^n$. Choose h so large that $\vec{y} \leq \vec{x} + h$, which we may always do (take $h = \mathsf{t}(\vec{y} - \vec{x})$, for instance), and let $\vec{z} = \vec{x} + h$. Then,

$$\begin{array}{rcll} \mathsf{t}(F(\vec{x}) - F(\vec{y})) + h & = & \mathsf{t}(F(\vec{x}) + h - F(\vec{y})) & \text{by (2)} \\ & = & \mathsf{t}(F(\vec{z}) - F(\vec{y})) & \text{by } \mathsf{H} \\ & \leq & \|F(\vec{z}) - F(\vec{y})\| & \text{by (2)} \\ & \leq & \|\vec{z} - \vec{y}\| & \text{by } \mathsf{N} \\ & = & \mathsf{t}(\vec{z} - \vec{y}) & \text{by choice of } h \\ & = & \mathsf{t}(\vec{x} - \vec{y}) + h & \text{by (2).} \end{array}$$

Subtracting h from the first and last formulae, we recover property T. Hence statement 4 holds and we have shown that 2 and 4 are equivalent.

QED

The equivalence of statements 1 and 2 in Proposition 1.1 is due to Crandall and Tartar, [2, Proposition 2].

Definition 1.1 *A function $F : \mathbf{R}^n \to \mathbf{R}^n$ is said to be topical if it satisfies any of the equivalent conditions of Proposition 1.1.*

It is helpful to know how to combine topical functions. If $F, G : \mathbf{R}^n \to \mathbf{R}^n$, then define $F \leq G$ if $F(\vec{x}) \leq G(\vec{x})$ for all $\vec{x} \in \mathbf{R}^n$. Functions form a lattice under this ordering, with $\vee$ and $\wedge$ defined pointwise. Let $F^-(\vec{x}) = -F(-\vec{x})$. The following elementary result follows easily from Proposition 1.1 and is left to the reader.

Lemma 1.1 *Let F, G be topical functions. Let $\lambda, \mu \in \mathbf{R}$ satisfy $\lambda, \mu \geq 0$ and $\lambda + \mu = 1$. Let $h \in \mathbf{R}$. The functions FG, $F \vee G$, $F \wedge G$, $F + h$, F^- and $\lambda F + \mu G$, are all topical.*

An interesting class of topical functions are min-max functions, [5, 6, 7], which provided the initial motivation for this study.

2 Weak convergence to the cycle time

In this section we begin the study of the cycle time vector: $\lim_{k\to\infty} F^k(\vec{x})/k$. Interest in this first arose in the study of min-max functions, [5], where, in certain applications, [4], it can be thought of naturally as an *"asymptotic average time to the next occurrence"*:

$$\lim_{k\to\infty} \frac{\sum_{i=1}^{k} F^i(\vec{x}) - F^{i-1}(\vec{x})}{k}.$$

The cycle time can be defined in greater generality than just for topical functions.

Lemma 2.1 *Let F be a nonexpansive function on a normed vector space V. If $\lim_{k\to\infty} F^k(\vec{x})/k$ exists at some point $\vec{x} \in V$, then it exists everywhere in V and has the same value.*

Proof: By nonexpansiveness, $\|F^k(\vec{x}) - F^k(\vec{y})\| \leq \|\vec{x} - \vec{y}\|$. The result follows.

QED

The notation $\chi(F)$ will denote the cycle time vector of F, when it exists. Note that $\chi(F) \in V$. The interest in χ stems from its connection with fixed points of F. If $F(\vec{x}) = \vec{x}$ then, clearly, $\chi(F)$ exists and $\chi(F) = 0$. Hence, if $\chi(F)$ does not exist, then F can cannot have a fixed point. Even if $\chi(F)$ does exist, it must be 0 in each component for F to have any chance of having a fixed point. Finally, we observe that $\chi(F)$ always provides a lower bound for how close F can come to having a fixed point.

Lemma 2.2 *Let F be a nonexpansive function on a normed vector space V. If $\chi(F)$ exists then $\|\chi(F)\| \leq \min_{\vec{x}\in V} \|F(\vec{x}) - \vec{x}\|$.*

Proof: Choose $\vec{x} \in V$. By nonexpansiveness,

$$\|F^k(\vec{x}) - \vec{x}\| = \| \textstyle\sum_{i=1}^k F^i(\vec{x}) - F^{i-1}(\vec{x})\| \leq k\|F(\vec{x}) - \vec{x}\|.$$

Hence, $\|\chi(F)\| \leq \|F(\vec{x}) - \vec{x}\|$. Since this holds for any $\vec{x} \in V$, the result follows.

QED

For topical functions, there is evidence that stronger results may hold. In particular, for min-max functions it is conjectured that $\chi(F)$ always exists and that if $\chi(F) = 0$ then F has a fixed point, [5]. (Both assertions would follow from the Duality Conjecture for min-max functions; see [5, 6] for more details.) For max-only functions, a sub-class of min-max functions which have been studied by many authors, [1, 3, 8, 9], both assertions are known to be true, [5, Propositions 2.1, 3.1]. It appears worthwhile, therefore, to study the existence of the cycle time vector in the general setting of topical functions. That is the purpose of the present paper.

When discussing sequences, we shall use the notation a^i rather than the more conventional a_i in order to avoid confusion with the components of a vector.

If a^i is a sequence of points in some metric space and f is any continuous function then $f(a^i)$ will converge, as $i \to \infty$, whenever a^i converges. If, on the other hand, a^i does not converge, it may still happen that $f(a^i)$ converges for certain functions f. We may call this, informally, "weak convergence".

Proposition 2.1 *Let F be a topical function and let $\vec{x} \in \mathbf{R}^n$. The sequences $\mathsf{t}(F^k(\vec{x})/k)$ and $\mathsf{b}(F^k(\vec{x})/k)$ both converge as $k \to \infty$ and both limits are independent of $\vec{x}$.*

Proof: Choose $\vec{x}, \vec{y} \in \mathbf{R}^n$. By (2) and property T, $F(\vec{x}) \leq F(\vec{y}) + \mathsf{t}(\vec{x} - \vec{y})$. By properties H and M, it then follows that $F^k(\vec{x}) \leq F^k(\vec{y}) + \mathsf{t}(\vec{x} - \vec{y})$. Hence, by (2),

$$\mathsf{t}(F^k(\vec{x})/k) \leq \mathsf{t}(F^k(\vec{y})/k) + \mathsf{t}(\vec{x} - \vec{y})/k.$$

By interchanging $\vec{x}$ and $\vec{y}$, it is easy to see that if $\lim_{k\to\infty} \mathsf{t}(F^k(\vec{x})/k)$ exists, then so must $\lim_{k\to\infty} \mathsf{t}(F^k(\vec{y})/k)$ and it must have the same value. A dual argument works for b in place of t. This gives the last assertion. It is now sufficent to consider $\vec{x} = 0$. By (2) we have that

$$\mathsf{b}(F^j(0)) \leq F^j(0) \leq \mathsf{t}(F^j(0)).$$

Applying F^i and using properties M and H, we see that

$$\mathsf{b}(F^j(0)) + F^i(0) \leq F^{i+j}(0) \leq F^i(0) + \mathsf{t}(F^j(0)). \tag{3}$$

Let $t^j = \mathsf{t}(F^j(0))$ and $b^j = \mathsf{b}(F^j(0))$. By (2), $t^{i+j} \leq t^i + t^j$ and $b^{i+j} \geq b^i + b^j$. Since $b^i \leq t^i$ for each i, it follows from the subadditive convergence theorem that t^i/i and b^i/i both converge.

QED

Definition 2.1 *If F is a topical function then $\overline{\chi}(F) = \lim_{k\to\infty} \mathsf{t}(F^k(0)/k)$ and $\underline{\chi}(F) = \lim_{k\to\infty} \mathsf{b}(F^k(0)/k)$.*

If $\chi(F)$ does exist then, by continuity of t and b, $\overline{\chi}(F) = \mathsf{t}(\chi(F))$ and $\underline{\chi}(F) = \mathsf{b}(\chi(F))$. But $\overline{\chi}$ and $\underline{\chi}$ give information even when χ does not exist. The following result merely elaborates on Lemma 2.2 and the proof is left to the reader.

Lemma 2.3 *If F is a topical function then*

$$\max_{\vec{x}\in\mathbf{R}^n} \mathsf{b}(F(\vec{x}) - \vec{x}) \;\leq\; \underline{\chi}(F) \;\leq\; \overline{\chi}(F) \;\leq\; \min_{\vec{x}\in\mathbf{R}^n} \mathsf{t}(F(\vec{x}) - \vec{x}).$$

It is interesting to ask whether the outermost inequalities are optimal. For certain min-max functions it can be shown that they collapse to equalities.

The existence of $\overline{\chi}$ and $\underline{\chi}$ appears to give no hint as to whether the underlying sequence $F^k(0)/k$ converges; weak convergence does not imply convergence. Nevertheless, $t^k = \mathsf{t}(F^k(0))$ must coincide with some component of the vector $F^k(0)$; similarly for $b^k = \mathsf{b}(F^k(0))$. From equation (3) we see that the r-th component of the vector sequence $F^k(0)$ satisfies

$$b^j + F^i(0)_r \leq F^{i+j}(0)_r \leq F^i(0)_r + t^j. \tag{4}$$

If $F^i(0)_r = t^i$ sufficiently often, this suggests that $F^i(0)_r/i$ should come close to $\overline{\chi}$ as $i \to \infty$. The question is, how often is "sufficiently often"? Clearly, there is at least one component r for which $F^i(0)_r = t^i$ infinitely often. If r is such a component, define the function $\phi : \mathbf{N} \to \mathbf{N}$ such that $\phi(k)$ gives the next index i, after k, at which that component coincides with t^i. In other words,

$$\phi(k) = \min_i \{ i \in \mathbf{N} \mid k < i,\ F^i(0)_r = t^i \}. \tag{5}$$

Proposition 2.2 *With the notation above, if $\phi(k)/k \to 1$ as $k \to \infty$ then*

$$\lim_{i\to\infty} F^i(0)_r/i = \overline{\chi}(F).$$

A similar statement holds for $\underline{\chi}$ in place of $\overline{\chi}$.

Proof: Let $a^i = F^i(0)_r$ and $\overline{\chi} = \overline{\chi}(F)$. We then have, using (4)

$$\begin{array}{rcl} a^i & \leq & t^i \\ a^{i+j} & \leq & a^i + t^j \\ t^{i+j} & \leq & t^i + t^j \end{array} \tag{6}$$

and $t^i/i \to \overline{\chi}$. We are required to show that $a^i/i \to \overline{\chi}$. Choose $\epsilon > 0$. Since $a^i/i \leq t^i/i$, it is enough to show that, for all sufficiently large i, $\overline{\chi} - \epsilon < a^i/i$. Suppose this is not the case and that i_k is a sequence such that $a^{i_k}/i_k \leq \overline{\chi} - \epsilon$. Let $u_k = \phi(i_k) - i_k$ and note that by (5), $u_k > 0$. Since $\phi(i)/i \to 1$, we may choose k so large that $u_k/i_k < \epsilon/2t^1$. Note further that, by (5), $a^{\phi(i_k)} = t^{\phi(i_k)}$. Hence, from equations (6) above, $t^{\phi(i_k)} \leq a^{i_k} + t^{u_k}$. We may rewrite this as

$$
\begin{aligned}
\frac{t^{\phi(i_k)}}{\phi(i_k)} &\leq \left(\frac{a^{i_k}}{i_k}\right)\left(\frac{i_k}{\phi(i_k)}\right) + \left(\frac{t^{u_k}}{u_k}\right)\left(\frac{u_k}{\phi(i_k)}\right) \\
&\leq \overline{\chi} - \epsilon + t^1\left(\frac{u_k}{i_k}\right) \\
&\leq \overline{\chi} - \epsilon/2
\end{aligned}
$$

where we have used the estimate $t^{u_k} \leq u_k t^1$, which follows from subadditivity, and the fact that $i_k < \phi(i_k)$. However, we now see that for infinitely many i, $t^i/i \leq \overline{\chi} - \epsilon/2$. But this is impossible since $t^i/i \to \overline{\chi}$. Hence, a^i/i must converge to $\overline{\chi}$.

For the last assertion of the Proposition, let $a^i = F^i(0)_r$ where r is the component in question, which coincides with b^i sufficiently often. Using (4), we can write a similar set of equations to (6) above, but with the inequalities reversed. If we now multiply through by -1, we find ourselves in the situation discussed above and the assertion follows.

QED

This result does not seem to make full use of the fact that t^k must coincide with some component of $F^k(0)$. In particular, if one component coincides with t very infrequently, the other components must coincide more frequently in order to compensate. One feels that a stronger result ought to hold. The following conjecture is consistent with all the evidence at our disposal.

Conjecture 2.1 *Let $F : \mathbf{R}^n \to \mathbf{R}^n$ be a topical function. There exist components s, r, where $1 \leq s, r \leq n$, such that*

$$
\begin{aligned}
\lim_{i\to\infty} F^i(0)_r/i &= \overline{\chi}(F) \\
\lim_{i\to\infty} F^i(0)_s/i &= \underline{\chi}(F).
\end{aligned}
$$

3 The cycle time in low dimensions

When $n = 1$, any function satisfying property H must have the form $F(x) = x + a$, for some a. All such functions satisfy property M. Clearly, $\chi(F) = a$.

Now suppose $n = 2$. Property H allows us to reduce the effective dimension of a topical function by 1. When the initial dimension is 2, this technique can be very effective, [7, §3].

Let $F : \mathbf{R}^n \to \mathbf{R}^n$ satisfy property H. Following [7], we introduce the auxiliary function, $H : \mathbf{R}^{n-1} \to \mathbf{R}^{n-1}$, associated to F. Let $\pi : \mathbf{R}^n \to \mathbf{R}^{n-1}$ be defined by $\pi(x_1, \cdots, x_n) = (x_1 - x_n, \cdots, x_{n-1} - x_n)$. It is helpful to think of π as projection parallel to the diagonal onto the hyperplane $x_n = 0$. Now define H by $H(\vec{x}) = \pi(F(\vec{x}, 0))$ for any $\vec{x} \in \mathbf{R}^{n-1}$. The dynamic behaviour of F is mirrored exactly by H.

Lemma 3.1 (Compare [7, Lemma 3.1].) *For any $\vec{x} \in \mathbf{R}^n$, $H^i(\pi(\vec{x})) = \pi(F^i(\vec{x}))$.*

Proof: Let $\vec{x} \in \mathbf{R}^n$. Note that $\pi(\vec{x}+h) = \pi(\vec{x})$ and that $(\pi(\vec{x}), 0) = \vec{x} - x_n$. Hence, by property H, $H(\pi(\vec{x})) = \pi(F(\pi(\vec{x}), 0)) = \pi(F(\vec{x}) - x_n) = \pi(F(\vec{x}))$. In other words, the following diagram commutes:

$$\begin{array}{ccc} \mathbf{R}^n & \xrightarrow{F} & \mathbf{R}^n \\ \pi \downarrow & & \downarrow \pi \\ \mathbf{R}^{n-1} & \xrightarrow{H} & \mathbf{R}^{n-1} \end{array} .$$

The result follows immediately, upon iterating this diagram.

QED

There are other projections and corresponding auxiliary functions which could be defined, for which an analogous result would hold, but this is sufficient for our purposes here.

Proposition 3.1 *If $F : \mathbf{R}^2 \to \mathbf{R}^2$ is a topical function in dimension 2, then $\chi(F)$ exists.*

Proof: Let H be the auxiliary function of F as above. If $H^i(0)$ is not a montonic sequence then, by continuity of H, there must exist $x \in \mathbf{R}$ where $H(x) = x$. Hence $F(x, 0) = (x, 0) + F_2(x, 0)$ and by repeated use of H, we see that $\chi(F)$ exists with $\chi(F) = F_2(x, 0)$. Hence we may assume that $H^i(0)$ is monotonic. Without loss of generality, assume that $H^i(0) \leq H^{i+1}(0)$ for $i \geq 0$. In particular, $0 \leq H^i(0)$. But then, by Lemma 3.1, $F^i(0,0)_2 \leq F^i(0,0)_1$ for all i. In other words, $\mathsf{t}(F^i(0,0)) = F^i(0,0)_1$ and $\mathsf{b}(F^i(0,0)) = F^i(0,0)_2$. The result follows immediately from Proposition 2.1.

QED

When $n = 3$, the dynamic behaviour of F is more complicated.

Theorem 3.1 *Let $\{a^i\}$, $i \geq 1$, be any sequence of real numbers drawn from the unit interval $[0, 1]$. There exists a topical function $F : \mathbf{R}^3 \to \mathbf{R}^3$, such that $F^i(0,0,0)_2 = a^1 + \cdots + a^i$.*

The proof requires some preparation. A function $F : \mathbf{R}^n \to \mathbf{R}^n$ can be decomposed into n component functions $F_i : \mathbf{R}^n \to \mathbf{R}$. It is easy to see that F satisfies property M if, and only if, each F_i does, and similarly for property H. We will consider functions $F : \mathbf{R}^3 \to \mathbf{R}^3$ for which two of the components have a simple form:

$$F(x, y, z) = (x, f(x, y, z), z + 1). \tag{7}$$

To ease the notation, we use (x, y, z) for coordiantes in $\mathbf{R}^3$.

Let $g_z(y) = g(y, z) = f(0, y, z)$. It is then easy to see that $F^i(0,0,0) = (0, c^i, i)$ where $c^i = g_{i-1}(c^{i-1})$ and $c^0 = 0$. We shall construct $g : \mathbf{R}^2 \to \mathbf{R}$ so that

$$g_{i-1}(g_{i-2}(...(g_0(c^0)))) = a^1 + \cdots a^i. \tag{8}$$

We shall then have to show that the corresponding $f : \mathbf{R}^3 \to \mathbf{R}$ satisfies properties M and H. The following property of functions $f : \mathbf{R}^n \to \mathbf{R}$ is crucial.

- If $h \geq 0$, $f(\vec{x} + h) \leq f(\vec{x}) + h$ (sub-homogeneity) SH

Lemma 3.2 *Let $f : \mathbf{R}^n \to R$ be given and define $g : \mathbf{R}^{n-1} \to \mathbf{R}$ by $g(x_1, \cdots, x_{n-1}) = f(0, x_2, \cdots, x_n)$. If f satisfies* M *and* H *then g satisfies* M *and* SH. *Conversely, if $g : \mathbf{R}^{n-1} \to \mathbf{R}$ is given, define $f : \mathbf{R}^n \to \mathbf{R}$ by $f(x_1, \cdots, x_n) = g(x_2 - x_1, \cdots, x_n - x_1) + x_1$. If g satisfies* M *and* SH *then f satisfies* M *and* H.

Proof: Suppose f is given and satisfies M and H. Since g is the restriction of f to the hyperplane $x_1 = 0$, g clearly satisfies M. Choose $\vec{x} \in \mathbf{R}^{n-1}$ and $h \geq 0$. Evidently, $(-h, \vec{x}) \leq (0, \vec{x})$ in $\mathbf{R}^n$. By property M for f, $f(-h, \vec{x}) \leq f(0, \vec{x})$. By property H for f, this can be rewritten as $g(\vec{x} + h) \leq g(\vec{x}) + h$, as required.

Now suppose that g is given and satisfies M and SH. By construction, it is immediate that f satisfies H. So suppose that $\vec{x} \leq \vec{y}$ in $\mathbf{R}^n$. Then

$$\begin{array}{rcll} (x_2 - x_1, \cdots, x_n - x_1) & \leq & (y_2 - x_1, \cdots, y_n - x_1) & \\ g(x_2 - x_1, \cdots, x_n - x_1) & \leq & g(y_2 - x_1, \cdots, y_n - x_1) & \text{by M} \\ & = & g(y_2 - y_1 + y_1 - x_1, \cdots, y_n - y_1 + y_1 - x_1) & \\ & \leq & g(y_2 - y_1, \cdots, y_n - y_1) + y_1 - x_1 & \text{by SH, } x_1 \leq y_1 \end{array}$$

Hence, $g(x_2 - x_1, \cdots, x_n - x_1) + x_1 \leq g(y_2 - y_1, \cdots, y_n - y_1) + y_1$. In other words, $f(\vec{x}) \leq f(\vec{y})$ so that f satisfies M.

QED

The proof of Theorem 3.1 is based on the following method of constructing 2-dimensional functions out of 1-dimensional functions. Let $f, h : \mathbf{R} \to \mathbf{R}$. Define $g : \mathbf{R}^2 \to \mathbf{R}$ by

$$g(y, z) = f(y - h(z)) + h(z). \tag{9}$$

The graph of $g(y, z)$ as a function of y, for fixed z, is obtained from the graph of f by sliding in a direction parallel to the diagonal. The extent of the "slide" is determined by $h(z)$.

Lemma 3.3 *Suppose that f satisfies property* SH. *Let $h(z') - h(z) = u$ and $y' - y = v$. If $v - u \geq 0$, then $g(y', z') - g(y, z) \leq v$.*

Proof:

$$\begin{array}{rcll} y' - h(z') & = & y - h(z) + v - u & \\ f(y' - h(z')) & = & f(y - h(z) + v - u) & \\ & \leq & f(y - h(z)) + v - u & \text{by SH} \\ f(y' - h(z')) + h(z') & \leq & f(y - h(z)) + h(z) + v & \\ g(y', z') & \leq & g(y, z) + v\ . & \end{array}$$

QED

Lemma 3.4 *If $f, h : \mathbf{R} \to \mathbf{R}$ both satisfy properties* M *and* SH *then so does g.*

Proof: If $(z, y) \leq (z, y')$, then $g(z, y) \leq g(z, y')$ because f satisfies M. If $(z, y) \leq (z', y)$, then by M for h, $h(z) - h(z') \leq 0$. So, in the notation of Lemma 3.3, $u \leq 0$ and $v = 0$. Hence, $v - u \geq 0$ and by Lemma 3.3, $g(z, y) - g(z', y) \leq 0$. In other words, $g(z, y) \leq g(z', y)$. It now follows immediately that g satisfies property M.

Now suppose that $(z', y') = (z, y) + h$, where $h \geq 0$. By SH for h, $h(z') - h(z) \leq h$ and therefore $v - u \geq 0$. Hence, by the claim above, $g(y', z') \leq g(y, z) + h$, so that g satisfies SH.

QED

We can now complete the proof of Theorem 3.1.

Proof: Define f by

$$f(y) = \begin{cases} y+1 & \text{if } y \leq -1 \\ 0 & \text{if } -1 \leq y \leq 0 \\ y & \text{if } y \geq 0 \end{cases} . \tag{10}$$

It is easy to see that f satisfies both M and SH.

It will be convenient to use the notation $s^i = a^1 + \cdots a^i$; by convention, $s^0 = 0$. Recall that for $z \in \mathbf{R}$, $z = [z] + \{z\}$, where $[z] \in \mathbf{Z}$ and $0 \leq \{z\} < 1$. Now define h by

$$h(z) = \begin{cases} z+1 & \text{if } z \leq -1 \\ s^{[z]+1} + \{z\} a^{[z]+2} & \text{if } z \geq -1 \end{cases} . \tag{11}$$

It is clear that both definitions agree that $h(-1) = 0$, so that h is well defined. To show that h satisfies M, it is sufficient to deal separately with the cases $z \leq z' \leq -1$ and $-1 \leq z \leq z'$. The former is immediate and the latter follows easily from the fact that $a^i \geq 0$ for $i \geq 1$. As for property SH, it suffices to observe that the graph of h is piecewise linear and the linear segments have slopes lying between 0 and 1.

If we now build g according to (9) then, by Lemma 3.4, g satisfies M and SH.

We claim that $g_{i-1}(g_{i-2}(...(g_0(0)))) = s^i$ for $i \geq 1$. The proof is by induction on i. For $i = 1$, it follows from (11) that $h(0) = s^1 = a^1$. Since, $-1 \leq -a^1 \leq 0$, it follows from (10) that $f(-h(0)) = 0$. Hence, by (9), $g(0, 0) = s^1$. Now assume that $g_{k-1}(g_{k-2}(...(g_0(0)))) = s^k$ for $k > 1$. Then, by (9), $g_k(s^k) = g(s^k, k) = f(s^k - h(k)) + h(k)$. Now, by (11), $h(k) = s^{k+1}$. Since $s^k - s^{k+1} = -a^{k+1}$ and $-1 \leq -a^{k+1} \leq 0$ by choice of the sequence $\{a^i\}$, it follows that $f(s^k - h(k)) = 0$. Hence, $g(s^k, k) = s^{k+1}$. The claim follows by induction.

Now construct $f : \mathbf{R}^3 \to \mathbf{R}$ following the prescription in Lemma 3.2. By Lemma 3.2, f satisfies M and H. Hence the function F given by (7) is a topical function with $F^i(0,0,0)_2 = a^1 + \cdots a^i$.

QED

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